Application

Second Year

Exercises 1

Prove that

$$\boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}$$

$$\begin{aligned}
L.H.S = \cos 4\theta &= \cos^2(2\theta) - \sin^2(2\theta) \\
&= (\cos^2\theta - \sin^2\theta)^2 - (2\sin\theta\cos\theta)^2 \\
&= \cos^4\theta - 2\sin^2\theta\cos^2\theta + \sin^4\theta - 4\sin^2\theta\cos^2\theta \\
&= \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\
&= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1 = R.H.S
\end{aligned}$$

Exercises 2

If $z = x + jy$, find the equation of the locus $\arg(z^2) = \frac{\pi}{4}$.

let $Z = x + iy$, $Z^2 = x^2 - y^2 + i(2xy)$

$$\arg(z^2) = \tan^{-1}\left(\frac{x^2-y^2}{2xy}\right) = \frac{\pi}{4}$$

$$\therefore \frac{x^2-y^2}{2xy} = 1 \qquad x^2 - y^2 = 2xy$$

$$\therefore x^2 - 2xy - y^2 = 0$$

Exercises 3

1-Expand $\sin 4\theta$ in powers of $\sin\theta$ and $\cos\theta$.

2- Express $\cos^4\theta$ in terms of cosines of multiples of θ.

3- If $z = x + jy$, find the equations of the two loci defined by

(a) $|z-4| = 3$ (b) $\arg(z+2) = \frac{\pi}{6}$

$\boxed{1}$ $\sin 4\theta = \text{Img}\left[(e^{i\theta})^4\right] = \text{Img}\left[(\cos\theta + i\sin\theta)^4\right]$

$= \text{Img}\left[\cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta\right]$

$= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\boxed{2}$ $\cos^4\theta = Re\left[(e^{i\theta})^4\right] = Re\left[e^{i4\theta}\right]$

$= Re\left[\cos 4\theta + i\sin 4\theta\right] = \cos 4\theta$

$\boxed{3}$ (a) $|z-4| = 3$ let $z = x+iy$

$|x+iy-4| = 3$ $\therefore \sqrt{(x-4)^2 + y^2} = 3$

$(x-4)^2 + y^2 = 9$

(b) $\arg(z+2) = \frac{\pi}{6}$ $\therefore \arg(x+2+iy) = \frac{\pi}{6}$

$\tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{6}$

$\therefore \frac{y}{x+2} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

$\therefore y = \frac{\sqrt{3}}{3}(x+2)$

Exercises 4

Show that $u(x, y) = x^3y - y^3x$ is an harmonic function and find the function $v(x, y)$ that ensures that $f(z) = u(x, u) + jv(x, y)$ is analytic. That is, find the function $v(x, y)$ that is conjugate to $u(x, y)$.

$u_x = 3x^2y - y^3 \qquad u_{xx} = 6xy$

$u_y = x^3 - 3y^2x \qquad u_{yy} = -6xy$

$\therefore u_{xx} + u_{yy} = 6xy - 6xy = \text{zero}$

$\therefore u(x,y)$ is an harmonic function

$u_x = V_y \quad , \quad u_y = -V_x$

$V_x = -u_y = 3xy^2 - x^3 \qquad \therefore V = \frac{3}{2}x^2y^2 - \frac{1}{4}x^3 + g(y$

$V_y = 3x^2y + g'(y) = u_x = 3x^2y - y^3$

$\therefore g'(y) = -y^3 \qquad \therefore g(y) = -\frac{1}{4}y^4 + K$

$\therefore V = \frac{3}{2}x^2y^2 - \frac{1}{4}x^3 - \frac{1}{4}y^4 + K \Longrightarrow \#$

$\therefore f(z) = (x^3y - y^3x) + i\left(\frac{3}{2}x^2y^2 - \frac{1}{4}x^3 - \frac{1}{4}y^4 + K\right)$

Exercises 5 (Harmonic functions)

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

1. $u = e^{-x} \sin 2y$
2. $u = xy$
3. $v = xy$
4. $v = -y/(x^2 + y^2)$
5. $u = \ln |z|$
6. $v = \ln |z|$
7. $u = x^3 - 3xy^2$
8. $u = 1/(x^2 + y^2)$
9. $v = (x^2 - y^2)^2$

$\boxed{1}$ $u = e^{-x} \sin 2y$

$u_x = -e^{-x} \sin 2y$, $u_{xx} = e^{-x} \sin 2y$

$u_y = 2e^{-x} \cos 2y$, $u_{yy} = -4e^{-x} \sin 2y$

$u_{xx} + u_{yy} \neq 0$ u isn't harmonic

$\boxed{2}$ $u = xy$

$u_x = y$ $\qquad u_{xx} = 0$

$u_y = x$ $\qquad u_{yy} = 0$

$u_{xx} + u_{yy} = 0$ $\therefore$ u is harmonic

$u_x = v_y$, $u_y = -v_x$

$v_y = y$ $\quad \therefore V = \frac{1}{2} y^2 + g(x)$

$v_x = -u_y = g'(x) = -x$

$g(x) = -\frac{1}{2} x^2 + k$

$\therefore V = \frac{1}{2} y^2 - \frac{1}{2} x^2 + k$

$\boxed{4}$ $u = -y/(x^2+y^2)$

$u_x = \frac{-y(2x)}{(x^2+y^2)^2}$, $u_y = \frac{-(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$

$u_{xx} = \frac{-2y(x^2+y^2) + 2(x^2+y^2)2x - 2xy}{(x^2+y^2)^4}$

$u_{yy} = \frac{(x^2+y^2)^2(-2y-4y) - (-(x^2+y^2) - 2y^2) * 4y(x^2+y^2)}{(x^2+y^2)^4}$

$\boxed{5}$ $u = \ln|z| = \ln (x^2+y^2)^{\frac{1}{2}}$

$u_x = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)}$, $u_{xx} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}$

$u_y = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)}$, $u_{yy} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$

$u_{xx} + u_{yy} \neq 0$ $\quad \therefore$ u isn't harmonic

$\boxed{7}$ $u = x^3 - 3x^2$

$u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

$u_y = -6xy$, $u_{yy} = -6x$

$u_{xx} + u_{yy} = 0$ $\quad$ So u is harmonic

$u_x = v_y$ $\quad \therefore 3x^2 - 3y^2 = v_y$

$V = 3x^2 y - y^3 + g(x)$

$u_y = -v_x$

$\therefore 6xy = 6xy + g'(x)$ $\quad g'(x) = 0$ $\quad g(x) = k$

$\therefore V = 3x^2 y - y^3 + k$

Determine a, b, c such that the given functions are harmonic and find a harmonic conjugate.

1- $U = ax^2 + y^2$

2. $u = e^{3x} \cos ay$

3. $u = \sin x \cosh cy$

4. $u = ax^3 + by^3$

1 $u = ax^2 + y^2$

$u_x = 2ax$ $\quad u_{xx} = 2a$

$u_y = 2y$ $\quad u_{yy} = 2$

$u_{xx} + u_{yy} = 0 = 2a + 2$

$\boxed{\therefore\ a = -1}$

2 $u = e^{3x} \cos ay$

$u_x = 3e^{3x} \cos ay$ $\quad u_{xx} = 9e^{3x} \cos ay$

$u_y = -ae^{3x} \sin ay$ $\quad u_{yy} = -a^2 e^{3x} \cos ay$

$u_{yy} + u_{xx} = 0 = 9e^{3x} \cos ay - a^2 e^{3x} \cos ay$

$9\cancel{e^{3x}}\cancel{\cos ay} = a^2 \cancel{e^{3x}} \cancel{\cos ay}$

$a^2 = 9$ $\quad \boxed{a = \pm 3}$

3 $u = \sin x \cosh cy$

$u_x = \cos x \cosh cy$ $\quad u_{xx} = -\sin x \cosh cy$

$u_y = c \sin x \sinh yc$ $\quad u_{yy} = c^2 \sin x \cosh cy$

$u_{xx} + u_{yy} = 0 = -\sin x \cosh cy + c^2 \sin x \cosh cy$

$\cancel{\sin x}\ \cancel{\cosh cy} = c^2\ \cancel{\sin x}\ \cancel{\cosh cy}$

$c^2 = 1$ $\quad \boxed{c = \pm 1}$

4 $u = ax^3 + by^3$

$u_x = 3ax^2$ $\quad u_{xx} = 6ax$

$u_y = 3by^2$ $\quad u_{yy} = 6by$

$u_{xx} + u_{yy} = 0 = 6ax + 6by$

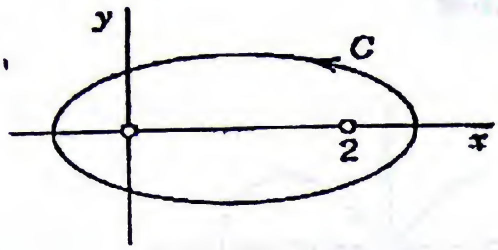
$\therefore\ ax + by = 0$

$\boxed{\therefore\ b = \frac{-ax}{y}}$

Exercises 7

Evaluate

$$\oint_C \frac{7z-6}{z^2-2z}\,dz,\ C \text{ as shown}$$



$\in$ Contour

$$z^2 - 2z = 0 \qquad z(z-2) = 0 \qquad z=0\ ,\ z=2$$

$$\oint_C \frac{7z-6}{z(z-2)}\,dz$$

$$I = \oint \frac{(7z-6)/z}{z-2}\,dz + \oint \frac{(7z-6)/(z-2)}{z}\,dz$$

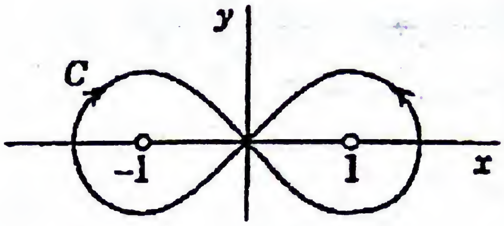
$$= 2\pi i \left[\frac{7z-6}{z}\right]_{z=2} + 2\pi i \left[\frac{7z-6}{z-2}\right]_{z=0}$$

$$= 2\pi i \left(\underbrace{\frac{14-6}{2}}_{\to 4}\right) + 2\pi i \left(\underbrace{\frac{0-6}{0-2}}_{\to 3}\right)$$

$$= 8\pi i + 6\pi i = \boxed{14\pi i}$$

Exercises 8
Evaluate

$$\oint_C \frac{dz}{z^2 - 1}, \; C \text{ as shown}$$



$$\oint \frac{dz}{z^2-1} = \oint \frac{dz}{(z-1)(z+1)} \qquad z=-1,\ z=1 \in \text{contour}$$

$$I = \oint \frac{1/(z+1)}{(z-1)}\,dz + \oint \frac{1/(z-1)}{(z+1)}\,dz$$

$$= 2\pi i \left[\frac{1}{z+1}\right]_{z=1} - 2\pi i \left[\frac{1}{z-1}\right]_{z=-1}$$

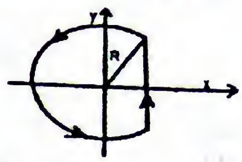
$$= \cancel{2}\pi i * \frac{1}{\cancel{2}} - \cancel{2}\pi i * \frac{-1}{\cancel{2}}$$

$$= \pi i + \pi i = \boxed{2\pi i}$$

Exercises 9

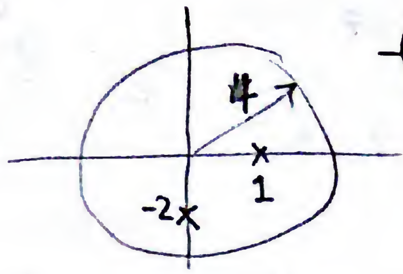
(a) **Evaluate** $\oint_C \frac{z}{(z-1)(z+2i)}dz$ **around** $C:|z|=4$.

(b) Using Bromwich contour



To find inverse Laplace transform of

$$F(s) = \frac{1}{(s-1)(s-2)}$$



$$I = \oint \frac{z}{(z-1)(z+2i)}\,dz \qquad z=1\ ,\ z=-2i \in \text{contour}$$

$$I = \oint \frac{z/(z-1)}{z+2i}\,dz + \oint \frac{z/(z+2i)}{z-1}\,dz$$

$$= 2\pi i \left[\frac{z}{z-1}\right]_{z=-2i} + 2\pi i \left[\frac{z}{z+2i}\right]_{z=1}$$

$$= 2\pi i \left(\frac{-2i}{-2i-1}\right) + 2\pi i \left(\frac{1}{1+2i}\right)$$

$$= \frac{-4\pi}{1+2i} + \frac{2\pi i}{1+2i} = \frac{2\pi(-4+i)}{1+2i}$$

$$= \frac{2\pi(-4+i)(1-2i)}{1+4}$$

$$= \frac{2\pi}{5}(-4+i+8i+2)$$

$$= \frac{2}{5}\pi(-2+9i)$$

Expand $\dfrac{e^{3z}}{(z-2)^4}$ in a Laurent series about the point $z = 2$ and determine the nature of the singularity at $z = 2$.

$$\frac{e^{3z}}{(z-2)^4} = \frac{e^{3z-6+6}}{(z-2)^4} = \frac{e^{6}\cdot e^{3(z-2)}}{(z-2)^4}$$

$$= \frac{e^6}{(z-2)^4}\left[1 + \frac{3(z-2)}{1!} + \frac{9(z-2)^2}{2!} + \text{——}\right]$$

$$= e^6\left[\frac{1}{(z-2)^4} + \frac{3}{(z-2)^3} + \frac{9/2}{(z-2)^2} + \frac{9/2}{z-2} + \frac{81}{4!} + \frac{3^5}{5!}(z-2) + \text{——}\right]$$

$$\lim_{z\to z_0} f(z) = \lim_{z\to 2}\frac{e^{3z}}{(z-2)^4} = \frac{e^6}{0} = \infty \quad \text{not Removable}$$

$$\lim_{z\to 2}(z-2)f(z) = \infty$$

$$\lim_{z\to 2}(z-2)^2 f(z) = \infty$$

$$\lim_{z\to 2}(z-2)^3 f(z) = \infty$$

$$\lim_{z\to 2}(z-2)^4 f(z) = e^6$$

$\therefore$ $z = 2$ is pole of order 4

$$\mathfrak{A}_A(x)=\frac{x}{x+1}\quad,\ \mathfrak{A}_B(x)=\frac{1}{x^2+10}\ ,\ \mathfrak{A}_C(x)=\frac{1}{10^x}$$

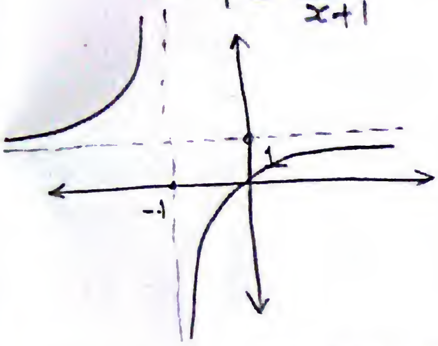
Determine mathematical membership functions graphs of the followings

a) $A\cup B$, $B\cap C$, b) $A\cup B\cup C$, $A\cap B\cap C$

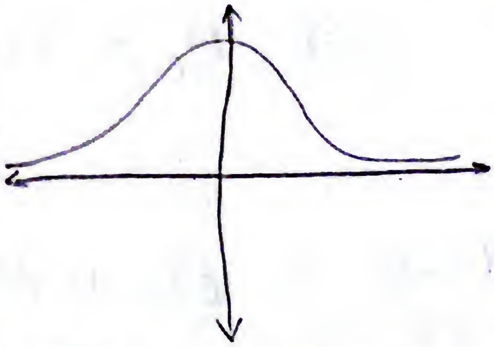
c) $A\cap C$, $B\cup C$ d) $A\cap B$, $A\cup B$

$\mu_A(x)=\frac{x}{x+1}=\frac{x+1-1}{x+1}$

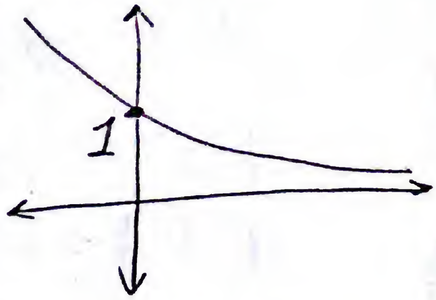
$=1-\frac{1}{x+1}$



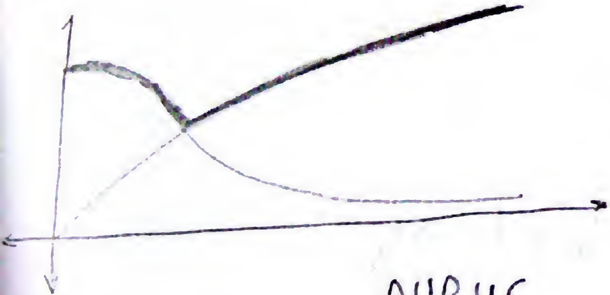
$\mu_B(x)=\frac{1}{x^2+10}$



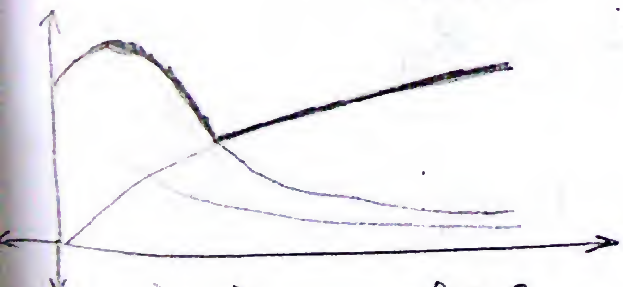
$\mu_c(x)=\frac{1}{10^x}$



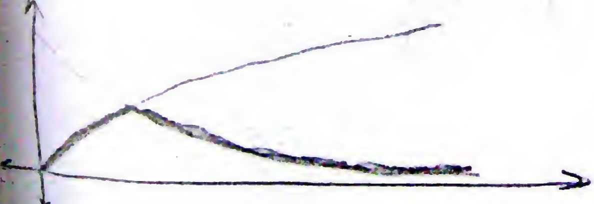
A∪B



A∪B∪C

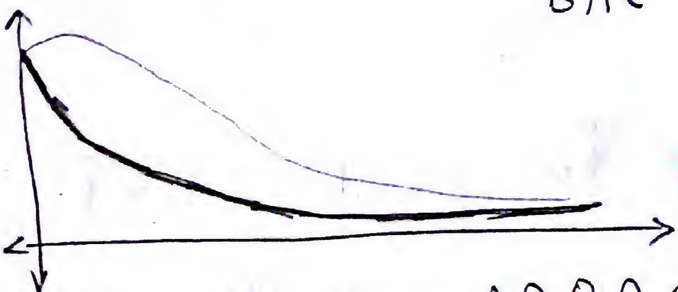


A∩C



A∩B

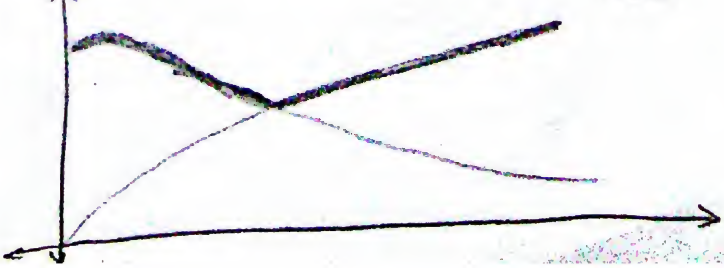
B∩C



A∩B∩C

B∪C

A∪B



Show the two fuzzy sets satisfy the De Morgan s Law ,

$$\mathfrak{A}_A = \frac{1}{1+(x-10)} \quad , \quad \mathfrak{A}_B(x) = \frac{1}{1+x^2}$$

$\mu_A > \mu_B$

➡ For $(A \cup B)' = A' \cap B'$

$1 - \max(\mu_A, \mu_B) = \min(1-\mu_A, 1-\mu_B)$

$\because \mu_A > \mu_B \qquad \therefore 1-\mu_A < 1-\mu_B$

$L.H.S = 1 - \max(\mu_A, \mu_B) = 1-\mu_A$

$R.H.S = \min(1-\mu_A, 1-\mu_B) = 1-\mu_A$

$\therefore R.H.S = L.H.S \Longrightarrow \#$

➡ For $(A \cap B)' = A' \cup B'$

$1 - \min(\mu_A, \mu_B) = \max(1-\mu_A, 1-\mu_B)$

$\because \mu_A > \mu_B \qquad \therefore 1-\mu_A < 1-\mu_B$

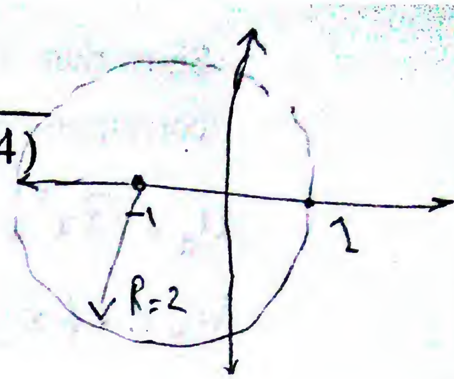
$L.H.S = 1 - \min(\mu_A, \mu_B) = 1-\mu_B$

$R.H.S = \max(1-\mu_A, 1-\mu_B) = 1-\mu_B$

$\therefore R.H.S = L.H.S$

Evaluate $\oint_c \frac{e^z}{z^2+1}dz$, $\oint_c \frac{\cos z\, dz}{z^2(z+2)}$, $\oint_c \frac{dz}{z^2(z+4)}$

where C is the circle $|z-1|=2$



$\boxed{1}$ $\oint_c \frac{e^z}{z^2+1} dz = \oint_c \frac{e^z}{(z+i)(z-i)} dz$

$\because z=i \ , z=-i \in$ Contour

$$\therefore I = \oint \frac{e^z/(z-i)}{(z+i)} dz + \oint \frac{e^z/(z+i)}{(z-i)} dz$$

$$= 2\pi i \left(\frac{e^z}{z-i}\right)_{z=-i} + 2\pi i \left(\frac{e^z}{z+i}\right)_{z=i}$$

$$= 2\pi i\left(\frac{e^{-i}}{-2i}\right) + 2\pi i\left(\frac{e^{i}}{2i}\right)$$

$$I = \frac{-\pi}{e^{i}} + \pi e^{i} = \boxed{\pi e^{i}(1-e^{-2i})} \Rightarrow \#$$

$\boxed{2}$ $\oint \frac{\cos z}{z^2(z+2)} dz$ $\qquad z=0 \ , z=-2 \in$ contour

$$I = \oint \frac{\cos z/z^2}{z+2} dz + \oint \frac{\cos z/(z+2)}{z^2} dz$$

$$= 2\pi i \left(\frac{\cos z}{z^2}\right)_{z=-2} + 2\pi i \left(\frac{\cos z}{z+2}\right)_{z=0} = \frac{\pi i}{2} \cos -2 + \pi i$$

$\boxed{3}$ $\oint \frac{dz}{z^2(z+4)}$ $\qquad z=-4 \notin$ contour , $z=0 \in$ contour

$$I = \oint \frac{1/(z+4)}{z^2} dz = \frac{2\pi i}{(2-1)!}\left(\frac{1}{z+4}\right)^{(2-1)}_{z=0} = 2\pi i \left(\frac{-1}{(z+4)^2}\right)_{z=0}$$

$$= \frac{-\pi i}{8}$$

Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2$ is a harmonic function and find corresponding analytic function $f(z) = u + iv$

$u_x = 3x^2 + 6x - 3y^2$, $u_{xx} = 6x + 6$

$u_y = -6xy - 6y$, $u_{yy} = -6x - 6$

$\therefore u_{xx} + u_{yy} = 0$ $\therefore u(x,y)$ is harmonic

$V_y = u_x$, $u_y = -V_x$

$V_y = 3x^2 - 3y^2 + 6x$ $V = 3x^2y - y^3 + 6xy + g(x)$

$V_x = 6xy + 6y + g'(x) = -u_y = 6xy + 6y$

$\therefore g'(x) = 0$ $\therefore g(x) = K$

$\therefore V = 3x^2y - y^3 + 6xy + k$

$\therefore f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 2) + i\,(3x^2y - y^3 + 6xy + k)$